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Path Integral Evaluation of the Eta Function

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In a three-dimensional model in which the gauge group does not commute with the Lorentz group, we demonstrate how the phase associated with the one-loop effective action (the η function) can be computed by considering a quantum mechanical path integral.

When one uses background field quantization (DeWitt, 1967; Abbott, 1981) the one-loop effective action $\Gamma^{(1)}$ is given by the logarithm of a functional determinant (DeWitt, 1967),

$$\Gamma^{(1)} = \ln \det{}^{-1/2} H \tag{1}$$

If H is not an elliptic operator, then (1) is replaced by

$$\Gamma^{(1)} = \ln \det{}^{-1/4} H^2 \tag{2}$$

In making this replacement, a phase may be lost, as H may have negative eigenvalues. This phase can be evaluated by determining $\eta(0)$, where $\eta(s)$ is defined by (Gilkey, 1984; Witten, 1989; McKeon and Sherry, n.d.)

$$\eta(s) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty dt\, t^{(s-1)/2} \operatorname{tr} H e^{-H^2 t} \tag{3}$$

If a parameter λ is inserted into H so that when $\lambda = 1$, the original function H is recovered, then (Gilkey, 1984)

$$\frac{d\eta_\lambda(s)}{d\lambda} = \frac{-s}{\Gamma((s+1)/2)} \int_0^\infty dt\, t^{(s-1)/2} \operatorname{tr}\left(\frac{dH_\lambda}{d\lambda} e^{-H_\lambda^2 t}\right) \tag{4}$$

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This expression has been used by Birmingham *et al.* (1990) to determine the η function in Chern–Simons theory,

In this paper we apply (4) to compute the η function in a model whose classical action is (McKeon, n.d.)

$$S_{cl}=\int d^3x\, \varepsilon_{\alpha\beta\gamma}(T_{\alpha\kappa}\,\partial_\beta T_{\gamma\kappa}-\tfrac{2}{3}\varepsilon_{\mu\nu\lambda}T_{\alpha\mu}T_{\beta\nu}T_{\gamma\lambda}) \tag{5}$$

[If the field $T_{\mu\nu}$ is decomposed

$$T_{\mu\nu}=\phi\delta_{\mu\nu}+A_\rho\varepsilon_{\mu\rho\nu}+\tau_{\mu\nu} \qquad (\tau_{\mu\nu}=\tau_{\nu\mu},\quad \tau_{\mu\mu}=0) \tag{6}$$

then the gauge transformation

$$\delta T_{\mu\nu}=(\partial_\mu\delta_{\nu\lambda}+2T_{\mu\rho}\varepsilon_{\nu\lambda\rho})\theta_\lambda\equiv D_{\mu\nu\lambda}(T)\theta_\lambda \tag{7}$$

under which (5) is invariant mixes ϕ, A_ρ, and $\tau_{\mu\nu}$, showing that the Lorentz group does not commute with the gauge symmetry in this model.]

We quantize by splitting $T_{\mu\nu}$ into a classical background $C_{\mu\nu}$ and a quantum part $Q_{\mu\nu}$ (DeWitt, 1967). With the gauge-fixing action

$$S_{\mathrm{gf}}=2\int d^3x\, B_\mu D_{\gamma\mu\delta}(C)Q_{\gamma\delta} \tag{8}$$

the bilinears in the effective action $S_{Cl}+S_{\mathrm{gf}}$ lead to a contribution to H in (1) given by

$$H_{\alpha\beta,\mu;\,\gamma\delta,\nu}=\begin{pmatrix}\varepsilon_{\alpha\rho\gamma}D_{\rho\beta\delta} & -D_{\alpha u v}\\ D_{\gamma\mu\delta} & 0\end{pmatrix} \tag{9}$$

so that

$$H_{\alpha\beta,\mu;\,\xi\zeta,\lambda}=\begin{pmatrix}-\delta_{\alpha\xi}D_{\rho\beta\delta}D_{\rho\delta\zeta}+2\varepsilon_{\beta\zeta\kappa}F_{\alpha\kappa\xi} & -\varepsilon_{\alpha\rho\gamma}\varepsilon_{\beta\lambda\kappa}F_{\gamma\kappa\rho}\\ \varepsilon_{\gamma\sigma\xi}\varepsilon_{\mu\zeta\kappa}F_{\sigma\kappa\gamma} & -D_{\gamma\mu\delta}D_{\gamma\delta\lambda}\end{pmatrix} \tag{10}$$

where

$$\begin{aligned}D_{\xi\beta\delta}D_{\alpha\delta\zeta}-D_{\alpha\beta\delta}D_{\xi\delta\zeta}&=2\varepsilon_{\beta\zeta\tau}(C_{\alpha\tau,\xi}-C_{\xi\tau,\alpha}+2\varepsilon_{\mu\nu\tau}C_{\alpha\mu}C_{\xi\nu})\\ &\equiv 2\varepsilon_{\beta\zeta\tau}F_{\alpha\tau\xi}\end{aligned} \tag{11}$$

In evaluating the right side of (4) in this model, we must compute the trace of the matrix element

$$\begin{aligned}M(x,y)&=\langle x|\,\frac{dH_\lambda}{d\lambda}e^{-H_\lambda^2 t}\,|y\rangle\\ &=\frac{dH_\lambda(x)}{d\lambda}\langle x|\,e^{-H_\lambda^2 t}\,|y\rangle\end{aligned} \tag{12}$$

We can now apply the quantum mechanical path integral (Itzykson and Zuber, 1980, p. 432)

$$F=\langle x|\exp\{-[\tfrac{1}{2}(\mathrm{p}-A(x))^2+V(x)]t\}|y\rangle$$
$$=\mathbb{P}\int Dq\exp\int_0^t d\tau\,[-\tfrac{1}{2}\dot{q}^2(\tau)+iA(q(\tau))\cdot\dot{q}(\tau)-V(q(\tau))] \qquad (13)$$
$$q(0)=y,\qquad q(t)=x,\qquad \mathbb{P}\text{ path ordering}$$

to determine this matrix element in much the same way as in McKeon (n.d.), Strassler (1992), and Polyakov (1987), where Green's functions were evaluated using this technique. The advantage of this approach is that no loop momentum integrals are encountered and that no three- or four-point couplings occur.

We start by making the expansion

$$F=\int Dq\exp\int_0^t d\tau\,[-\tfrac{1}{2}\dot{q}^2(\tau)]\sum_{N=0}^{\infty}\frac{1}{N!}\prod_{i=1}^{N}\int_0^t d\tau_i\,[iA(q(\tau_i))\cdot\dot{q}(\tau_i)-V(q(\tau_i))] \qquad (14)$$

We now make the plane wave decomposition

$$i\dot{q}(\tau_i)\cdot A(q(\tau_i))=i\dot{q}(\tau_i)\cdot\varepsilon^i\exp[ik^i\cdot q(\tau_i)]$$
$$\rightarrow\exp[i(\dot{q}(\tau_i)\cdot\varepsilon^i+k^i\cdot q(\tau_i))] \qquad (15a)$$
$$V(q(\tau_i))=v^i\exp[ik^i\cdot q(\tau_i)] \qquad (15b)$$

[In (15a) only terms linear in ε^i_μ are kept.] This reduces (14) to

$$F=\sum_{N=1}^{N}\frac{1}{N!}\int Dq\exp\int_0^t d\tau\left[-\frac{1}{2}\dot{q}(\tau)\right]\prod_{i=1}^{N}\int_0^t d\tau_i$$
$$\times\left\{\exp i\int_0^t d\tau\,[k^i\delta(\tau-\tau_i)-\varepsilon^i\delta(\dot{\tau}-\tau_i)]\cdot q(\tau)\right.$$
$$\left.-v^i\exp i\int_0^t d\tau\,[k^i\delta(\tau-\tau_i)]\cdot q(\tau)\right\} \qquad (16)$$

Each term in (16) can be evaluated using the standard D-dimensional result (Itzykson and Zuber, 1980, p. 432)

$$\int Dq\exp\int_0^t d\tau\left[-\frac{1}{2}\dot{q}^2(\tau)+\gamma(\tau)\cdot q(\tau)\right]=\frac{1}{(2\pi t)^{D/2}}\exp I(t) \qquad (17)$$

where

$$I(t)=\frac{-(x-y)^2}{2t}+\frac{1}{t}\int_0^t d\tau\,[x\tau+y(t-\tau)]\cdot\gamma(\tau)-\frac{1}{2}\int_0^t d\tau\,d\tau'\,\gamma(t)\cdot\gamma(\tau')\,G(\tau,\tau')$$

where

$$G(\tau,\tau')=\frac{1}{2}|\tau-\tau'|-\frac{1}{2}(\tau+\tau')+\frac{\tau\tau'}{t}$$

As we only need $\eta(0)$, it is necessary to consider only these terms in the expansion of (17) that give rise to poles in the integral over t in (4). It is easily shown that the only relevant term is then

$$f=-v'\int Dq\exp\int_0^t d\tau\,[-\tfrac{1}{2}\dot{q}^2(\tau)+ik'\delta(\tau-\tau'_1)\cdot q(\tau)] \tag{18}$$

If

$$\frac{dH_\lambda(x)}{d\lambda}=he^{ip\cdot x} \tag{19}$$

then we find by (12), (13), and (16)–(19) that

$$\begin{aligned}\operatorname{tr}\frac{dH_\lambda}{d\lambda}\exp(-H^2t)&=\int dx\,\operatorname{tr}(-hv)\exp[i(p+k')\cdot x]\\&\quad\times\int_0^t d\tau_1\exp-\frac{k'^2}{2}G(\tau_1,\tau_1)+\cdots\\&=-t\operatorname{tr}(hv)\frac{(2\pi)^3\,\delta(p+k')}{(2\pi t)^{3/2}}\\&\quad\times\int_0^1 du\exp-\frac{k'^2}{2}u(1-u)t+\cdots\end{aligned} \tag{20}$$

Substitution of (20) into (4) finally yields

$$\frac{d\eta_\lambda(0)}{d\lambda}=4\sqrt{2}\,\pi\operatorname{tr}(vh)\,\delta(p+k') \tag{21}$$

From (9) and (10) we see that

$$h=\frac{1}{\sqrt{2}}\begin{pmatrix}\varepsilon_{\alpha\rho\gamma}(2C_{\rho\kappa}\varepsilon_{\beta\delta\kappa}) & (-2C_{\alpha\kappa}\varepsilon_{\beta\nu\kappa})\\ (2C_{\gamma\kappa}\varepsilon_{\mu\delta\kappa}) & 0\end{pmatrix} \tag{22a}$$

and

$$v = \begin{pmatrix} 2\varepsilon_{\beta\zeta\kappa} F^{(\lambda)}_{\alpha\kappa\zeta} & -\varepsilon_{\alpha\rho\gamma}\varepsilon_{\beta\lambda\kappa} F^{(\lambda)}_{\gamma\kappa\rho} \\ \varepsilon_{\gamma\sigma\xi}\varepsilon_{\mu\zeta\kappa} F^{(\lambda)}_{\sigma\kappa\gamma} & 0 \end{pmatrix} \tag{22b}$$

where

$$F^{(\lambda)}_{\alpha\tau\xi} = \lambda C_{\alpha\tau,\xi} - \lambda C_{\xi\tau,\alpha} + 2\lambda^2 \varepsilon_{\mu\nu\tau} C_{\alpha\mu} C_{\xi\nu}$$

Together (21) and (22) yield

$$\eta(0) = -16\pi\varepsilon_{\alpha\rho\gamma} C_{\rho\kappa}(C_{\gamma\kappa,\alpha} + \tfrac{2}{3}\varepsilon_{\mu\nu\kappa} C_{\gamma\mu} C_{\alpha\nu})$$

We have thus established $\eta(0)$ in our model by using a quantum mechanical path integral.

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